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Waterbag reductions of the dispersionless discrete KP hierarchy

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Abstract. We consider the dispersionless limit of the discrete KP hierarchy. We describe a family of N-parameter 'waterbag' reductions of the hierarchy and give sufficient conditions for the reduced equations of motion to be hyperbolic. Using Geogdzhaev's method, we solve the initial value problem for the reduced system. The solution is implicit, in generalized hodograph form (Tsarev S P 1985 *Sov. Math.–Dokl.* **31** 488). The relationship of this form of the solution to the properties of the hierarchy, the reduction and the initial data is discussed briefly, indicating how this construction might be generalized.

1. Introduction

In recent years, numerous investigations have been made into integrable systems and their generalizations. One of the most studied is the discrete KP, or generalized Toda equations [2, 3, 12, 13]. This hierarchy [2, 3, 12] has the representation

$$\frac{\partial L}{\partial t_n} + [D_n, L] = 0 \qquad n = 1, 2, \ldots \tag{1}$$

where L and D_n are difference operators defined by

$$\begin{aligned} &L = \Delta + b_0 + b_1\Delta^{-1} + b_2\Delta^{-2} + \cdots \\ &\Delta = \exp\left(\frac{\partial}{\partial m}\right) \qquad D_n = \left(\frac{L^n}{n}\right)_+ . \end{aligned} \tag{2}$$

Here, Δ is the unit shift operator such that $\Delta g(m, t) = g(m+1, t)$, $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, while $(\,)_+$ denotes the polynomial part in Δ. The discrete KP equations are the consistency conditions for the following spectral equations:

$$L\varphi = \lambda\varphi \qquad \frac{\partial\varphi}{\partial t_n} = D_n\varphi. \tag{3}$$

This system is Hamiltonian and integrable. It also admits many different reductions. The usual Toda system is a good example of this [14, 15]; it is the t_1 flow in (1) with the restriction, invariant under the dynamics, that $b_i = 0$ for $i \geqslant 2$. There are many other types of reduction of the hierarchy (2), for instance with $L = \Delta + Q(1 - \Delta^{-1})^{-1}R$, where Q and R are row and column vectors. If Q, R are scalars, this is equivalent to the relativistic Toda chain [8]. In this paper we consider a particular limit of (1), the dispersionless limit or long-wave continuum limit, and a family of reductions of it. The outline of this paper is as follows. In section 2, we will discuss some properties of the dispersionless discrete KP hierarchy (dΔKP), and then its relation with an analogous system, the Benney hierarchy. In section 3, a family of reductions of this system is investigated in detail. Its initial value problem and solutions will be discussed in section 4.

2. The dispersionless discrete KP hierarchy

In order to obtain the dΔKP hierarchy, we take the formal semiclassical limit of (1). Under this limit, $\frac{\partial}{\partial m}$ is replaced by p and the commutator is replaced by the Poisson bracket with respect to p and x. Thus, the spectral problem becomes

$$\lambda = e^p + \sum_{k=0}^{\infty} B_k e^{-kp}. \tag{4}$$

The hierarchy (1) takes the form

$$\frac{\partial \lambda}{\partial t_n} + \{H_n, \lambda\}_{p,x} = 0 \qquad H_n = \left(\frac{\lambda^n}{n}\right)_+ \tag{5}$$

where $\{\ \}_{p,x}$ denotes the Poisson bracket with respect to p and x, and $(\ )_+$ denotes the part polynomial in e^p.

We can see that these continuum limits are always systems of hydrodynamic type, that is, systems of the form $(v_i)_t = M_i^j (v_j)_x$, where the M_i^j are functions of $\boldsymbol{v}$. Such systems with two dependent variables were solved by Riemann [17] using the hodograph transformation. Tsarev [16] showed how this result could be generalized to finitely many, N say, dependent variables; if a system of hydrodynamic type is Hamiltonian and is diagonalizable, it can be written as

$$\frac{\partial \lambda_i}{\partial t_n} + \mu_i \frac{\partial \lambda_i}{\partial x} = 0 \qquad i = 1, \ldots, N. \tag{6}$$

With these N *Riemann invariants* λ_i, and corresponding *characteristic speeds* μ_i, then it has a solution in the form

$$x - \mu_i t_n = w_i \qquad i = 1, \ldots, N. \tag{7}$$

The w_i are the characteristic speeds of any commuting flow of (6). These w_i satisfy a system of $N(N-1)$ overdetermined linear equations

$$\frac{\partial_i w_k}{w_i - w_k} = \Gamma_{ki}^k = \frac{\partial_i v_k}{v_i - v_k} \qquad \partial_i = \frac{\partial}{\partial \lambda_i} \qquad i \neq k. \tag{8}$$

Here the Γ_{ki}^k are the Christoffel symbols of the Riemannian metric associated with the Hamiltonian structure of (6). Provided that the system (6) is hyperbolic, so the speeds μ_i are real and distinct, and that it is not linearly degenerate (that is, $\partial \mu_i / \partial \lambda_i \neq 0$, $i = 1, \ldots, N$), then the solutions of (7) will satisfy (6), which is parametrized by the solutions of (8). These depend on N functions of one variable.

In this paper, we will look at a special reduction of the dΔKP hierarchy (5); by using the method introduced in [5] and the ideas in [9], we transform the reduced system to a diagonal system of hydrodynamic type, and then, by constructing a canonical transformation, find the solution of the initial value problem in the generalized hodograph form.

The dispersionless hierarchy (5) can be represented as a family of Vlasov equations

$$\frac{\partial f}{\partial t_n} + \{H_n, f\} = 0 \qquad f = f(x, p, \boldsymbol{t}) \qquad \boldsymbol{t} = (t_1, t_2, t_3, \ldots). \tag{9}$$

Here the moments B_n are defined by

$$B_n(x, \boldsymbol{t}) = \int_{-\infty}^{\infty} e^{np'} f(x, p', \boldsymbol{t})\, dp'.$$

It is simple to verify that the two constructions, (5) and (9), give the same system of partial differential equations for the moments B_n, for example,

$$\frac{\partial B_n}{\partial t_1} + \frac{\partial B_{n+1}}{\partial x} + nB_n\frac{\partial B_0}{\partial x} = 0 \qquad n = 0, 1, 2, \ldots. \tag{10}$$

These can be lifted to the simplest of (9)

$$\frac{\partial f}{\partial t_1} + \mathrm{e}^p\frac{\partial f}{\partial x} - \frac{\partial B_0}{\partial x}\frac{\partial f}{\partial p} = 0. \tag{11}$$

We now consider (4) as an asymptotic series, valid as $p \to +\infty$, of an analytic function, which we will also call $\lambda(p)$

$$\lambda(p) = \mathrm{e}^p + \int_{-\infty}^{\infty} \frac{\mathrm{e}^p f(x, p', \boldsymbol{t})}{\mathrm{e}^p - \mathrm{e}^{p'}}\,\mathrm{d}p'. \tag{12}$$

This function is analytic in the strips

$$J_k = \{p|\mathrm{Im}\,(p) \in (2k\pi, (2k+2)\pi), k \in \mathbb{Z}\}.$$

As $\lambda(p)$ is clearly periodic with period $2\pi\mathrm{i}$, we need to consider only the lower half of $J_0 : \{p \in \mathbb{C}|0 < \mathrm{Im}\,(p) \leqslant \pi\}$ and the upper half of $J_{-1} : \{p \in \mathbb{C}| -\pi \leqslant \mathrm{Im}\,(p) < 0\}$. We denote the restriction of λ to these two regions by λ_+ and λ_- respectively. The discontinuity across $\mathrm{Im}\,p = 0$ is given by $\lambda_+ - \lambda_- = -2\pi\mathrm{i}f$. Following [9, 10], with small modification of the contour, we obtain the boundary value of λ_+ as p approaches the real axis from above

$$\begin{aligned}\lambda_+(p) &= \mathrm{e}^p + \int_{\Lambda} \frac{\mathrm{e}^p f(x, p', \boldsymbol{t})}{\mathrm{e}^p - \mathrm{e}^{p'}}\,\mathrm{d}p' \\ &= \mathrm{e}^p + P\int \frac{\mathrm{e}^p f(x, p', \boldsymbol{t})}{\mathrm{e}^p - \mathrm{e}^{p'}}\,\mathrm{d}p' - \mathrm{i}\pi f\end{aligned} \tag{13}$$

where Λ is an indented contour passing below the point p and $P\int$ denotes the Cauchy principal value of the integral. We require that f should satisfy a Hölder condition, and that it tends to zero as $|p| \to \infty$ sufficiently rapidly. A straightforward calculation shows that if f satisfies (11) then $\lambda_+(p)$ satisfies

$$\frac{\partial \lambda_+}{\partial t_1} + \mathrm{e}^p\frac{\partial \lambda_+}{\partial x} = \frac{\partial \lambda_+}{\partial p}\left\{\frac{\partial p}{\partial t_1} + \mathrm{e}^p\frac{\partial p}{\partial x} + \frac{\partial B_0}{\partial x}\right\}.$$

It follows that the equation for the inverse function $p(\lambda_+, x, t)$ at constant λ_+ is

$$\frac{\partial p}{\partial t_1} + \frac{\partial}{\partial x}(\mathrm{e}^p + B_0) = 0$$

while, at constant p, we have

$$\frac{\partial \lambda_+}{\partial t_1} + \mathrm{e}^p\frac{\partial \lambda_+}{\partial x} - \frac{\partial B_0}{\partial x}\frac{\partial \lambda_+}{\partial p} = 0. \tag{14}$$

This is of course the t_1 flow in (5)†.

2.1. *The Benney hierarchy*

The Benney hierarchy [1] is constructed in a very similar way; its equations of motion can be obtained through

$$\frac{\partial \Lambda}{\partial T_n} + \left\{\left(\frac{\Lambda^n}{n}\right)_+, \Lambda\right\}_{P,X} = 0 \tag{15}$$

† We will use λ to denote λ_+ hereafter.

but with the generating function defined by

$$\Lambda(P) = P + \sum_{i=0}^{\infty} A_i P^{-(i+1)}.$$

The symbol $(\ )_+$ now denotes the polynomial part in P. A Vlasov equation [6], analogous to (11),

$$\frac{\partial f'}{\partial T_2} + P\frac{\partial f'}{\partial X} - \frac{\partial A_0}{\partial X}\frac{\partial f'}{\partial P} = 0 \qquad f' = f'(X, P, \boldsymbol{T}) \qquad \boldsymbol{T} = (T_1, T_2, T_3, \ldots) \tag{16}$$

gives rise to the moment equations

$$\frac{\partial A_n}{\partial T_2} + \frac{\partial A_{n+1}}{\partial X} + nA_{n-1}\frac{\partial A_0}{\partial X} = 0 \qquad n = 0, 1, 2, \ldots$$

where we have defined the moments to be $A_n = \int_{-\infty}^{\infty} P^n f' \, \mathrm{d}P$. The two systems are related due to the following theorem [11].

Theorem 2.1. *By setting* $\mathrm{e}^p + B_0 = P$, $t_1 = X$, $t_2 = -T_2$ *in the* t_1 *and* t_2 *flows of the* dΔKP *hierarchy*

$$\begin{aligned} &\frac{\partial f}{\partial t_1} + \mathrm{e}^p\frac{\partial f}{\partial x} - \frac{\partial B_0}{\partial x}\frac{\partial f}{\partial p} = 0 \\ &\frac{\partial f}{\partial t_2} + (\mathrm{e}^{2p} + B_0\mathrm{e}^p)\frac{\partial f}{\partial x} - \left(\mathrm{e}^p\frac{\partial B_0}{\partial x} + \frac{\partial B_1}{\partial x} + B_0\frac{\partial B_0}{\partial x}\right)\frac{\partial f}{\partial p} = 0 \end{aligned} \tag{17}$$

eliminating the x*-derivatives and setting* $f(x, p, \boldsymbol{t}) = f'(X, P, \boldsymbol{T})$, f' *will satisfy (16).*

Proof. We see that

$$A_0 = \int f' \, \mathrm{d}P = \int f' \frac{\mathrm{d}P}{\mathrm{d}p} \, \mathrm{d}p = \int f \mathrm{e}^p \, \mathrm{d}p = B_1.$$

By eliminating the x-derivative from equations (17), we have

$$\frac{\partial f}{\partial t_2} + (\mathrm{e}^p + B_0)\left(-\frac{\partial f}{\partial t_1} + \frac{\partial B_0}{\partial x}\frac{\partial f}{\partial p}\right) - \left(\mathrm{e}^p\frac{\partial B_0}{\partial x} + \frac{\partial B_1}{\partial x} + B_0\frac{\partial B_0}{\partial x}\right)\frac{\partial f}{\partial p} = 0.$$

Substituting $\mathrm{e}^p + B_0 = P$ in the above, we have a simpler equation

$$\frac{\partial f}{\partial t_2} - P\frac{\partial f}{\partial t_1} - \frac{\partial B_1}{\partial x}\frac{\partial f}{\partial p} = 0. \tag{18}$$

Now, using the fact that

$$\begin{aligned} \left.\frac{\partial f}{\partial t_2}\right|_p &= -\left.\frac{\partial f'}{\partial T_2}\right|_P + \frac{\partial f'}{\partial P}\frac{\partial B_0}{\partial t_2} \\ \left.\frac{\partial f}{\partial t_1}\right|_p &= \left.\frac{\partial f'}{\partial X}\right|_P + \frac{\partial f'}{\partial P}\frac{\partial B_0}{\partial t_1} \\ \left.\frac{\partial f}{\partial p}\right|_{t_2,t_1} &= \left.\frac{\partial f'}{\partial P}\right|_{T,X}(P - B_0) \end{aligned}$$

and eliminating x-derivatives of the moments B_i by using the moment equations

$$\begin{aligned} &\frac{\partial B_0}{\partial t_1} + \frac{\partial B_1}{\partial x} = 0 \\ &\frac{\partial B_1}{\partial t_1} + \frac{\partial B_2}{\partial x} + B_1\frac{\partial B_0}{\partial x} = 0 \\ &\frac{\partial B_0}{\partial t_2} + \frac{\partial B_2}{\partial x} + B_0\frac{\partial B_1}{\partial x} + B_1\frac{\partial B_0}{\partial x} = 0 \end{aligned}$$

in the equation (18), the result follows. □

Reductions of the hierarchy (5) can be classified in the same way as reductions of hierarchy (15) [9]. The reduction will again correspond to a univalent conformal mapping of a fixed domain, such as the half-plane, into a domain with N slits, and the Riemann invariants are the end points of the slits. These reduced systems are all semi-Hamiltonian, so can be solved by the hodograph transformation. In fact it is possible to use the approach of Geogdzhaev to solve them explicitly; we aim to illustrate this by looking at one important example of this in detail.

3. The 'waterbag' reduction

One important reduction is the so-called 'waterbag' model; this reduction is valid for any Vlasov equation. In the simplest case, the distribution function is taken to be piecewise constant on same region of phase space. The region is advected along the characteristics of the Vlasov equation, which are the solutions of Hamilton's equations. The area of the region is thus constant, and the region moves like a bag full of incompressible fluid; hence the name. It is of course sufficient to follow the boundary of the region, as the distribution remains constant inside it. Thus, we set, for $k = 1, \ldots, N$,

$$f = \begin{cases} F_k & p \in (p_k(x,t), p_{k+1}(x,t)) \\ 0 & \text{elsewhere} \end{cases}$$

where $p_k \in \mathbb{R}, \forall x, t$ with $p_i < p_j$ for distinct $i < j$; and F_k are constants. On substituting this ansatz into (11), we see that f satisfies (11) if p_k satisfies

$$\frac{\partial p_j}{\partial t} + \mathrm{e}^{p_j}\frac{\partial p_j}{\partial x} + \sum_{k=1}^{N} F_k \frac{\partial}{\partial x}(p_{(k+1,k)}) = 0 \qquad p_{(k+1,k)} = p_{k+1} - p_k. \tag{19}$$

Note the moments

$$B_0 = \sum_{k=1}^{N} F_k p_{(k+1,k)}$$

and

$$B_m = \sum_{k=1}^{N} \frac{1}{m} F_k (\mathrm{e}^{mp_{k+1}} - \mathrm{e}^{mp_k})$$

for $m \in \mathbb{N}, m \neq 0$ here. The generating function is of the form

$$\lambda(p) = \mathrm{e}^p + \sum_{k=1}^{N} F_k \int_{\tilde{\Lambda}_k} \frac{\mathrm{e}^p}{\mathrm{e}^p - \mathrm{e}^{p'}}\,\mathrm{d}p'$$

where $\tilde{\Lambda}_k, k = 1, \ldots, N$ are the segments of real axis with an indentation below the point p. This, after integration, is

$$\begin{aligned} &\lambda(p) = \mathrm{e}^p + \sum_{k=1}^{N} F_{(k,k-1)} \ln(\mathrm{e}^{-p} - \mathrm{e}^{-p_k}) - F_N \ln(\mathrm{e}^{-p} - \mathrm{e}^{-p_{N+1}}) \\ &F_0 = 0 \qquad F_{(k,k-1)} = F_k - F_{k-1}. \end{aligned} \tag{20}$$

Its branch cuts are taken along the real intervals

$$\begin{array}{lll} [p_{2i-1}, p_{2i}] \cup [p_{N+1}, \infty) & i = 1, \ldots, N/2 & N \text{ even} \\ [p_{2i-1}, p_{2i}] & i = 1, \ldots, (N+1)/2 & N \text{ odd.} \end{array}$$

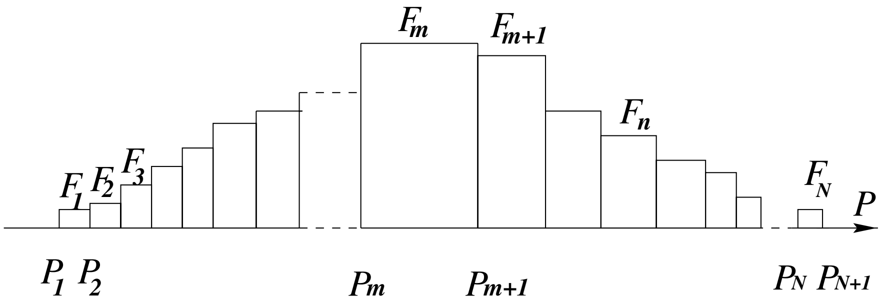

Figure 1. Sketch graph of f.

λ has asymptotics (4) as $p \to +\infty$. In addition, $\lambda(p)$ is periodic of period $2\pi\mathrm{i}$. To reduce the system to Riemann invariant form, we look at

$$\frac{\partial\lambda}{\partial t_n} + \frac{\partial H_n}{\partial p}\frac{\partial\lambda}{\partial x} - \frac{\partial H_n}{\partial x}\frac{\partial\lambda}{\partial p} = 0.$$

We see that at any point $(\hat{p}_i, \hat{\lambda}_i)$, where $\frac{\partial\lambda}{\partial p}|_{p=\hat{p}_i} = 0$, the equation can be written as

$$\frac{\partial\lambda(\hat{p}_i)}{\partial t_n} + \mu_n(\hat{p}_i)\frac{\partial\lambda(\hat{p}_i)}{\partial x} = 0 \qquad \mu_n(p) = \frac{\partial H_n}{\partial p}. \tag{21}$$

Provided that the number of parameters on which $\lambda(p)$ depends is the same as the number of these Riemann invariants, this may then be solved by the hodograph transformation [16, 17]. The characteristic speeds of this system are $\mu_n(\hat{p}_i)$ and Riemann invariants $\lambda(\hat{p}_i)$. We call the $\hat{p}_i$ the *characteristic momenta.*

Here, in the case of the waterbag reduction, for $\frac{\partial\lambda}{\partial p}$ to have $N+1$ distinct real roots $\hat{p}_i$, we assume that all $F_i \geqslant 0$ for $i = 0, \ldots, N-2$ and for the first m of $0 < m < n < N$, F_k are increasing and then decreasing for $k \in [n, N]$ as in figure 1. The Riemann invariants $\hat{\lambda}_i$ are the values of λ at the points $\hat{p}_i$.

4. The initial value problem

In the previous section, it has been shown that the equations can be reduced to a diagonal system of hydrodynamic type, by considering the points $p = \hat{p}_i$ where $\frac{\partial\lambda}{\partial p}|_{p=\hat{p}_i} = 0$. For this system to be hyperbolic, the characteristic speeds must be real and distinct. Since the characteristic speeds are polynomials in $\mathrm{e}^{\hat{p}_i}$ and moments B_k, and as the B_k are real, it sufficient to impose the condition that $\mathrm{e}^{\hat{p}_i}$ must be real for all i.

Consider the map $\lambda(p, x)$, defined by

$$\lambda(p, x) = \mathrm{e}^p + \sum_{i=1}^{N} F_{(k,k-1)} \ln(\mathrm{e}^{-p} - \mathrm{e}^{-p_k}) - F_N \ln(\mathrm{e}^{-p} - \mathrm{e}^{-p_{N+1}}).$$

It is periodic of period $2\pi\mathrm{i}$ and analytic in a region J_k of the p-plane, bounded by the straight lines $\mathrm{Im}\,(p) = 2k\pi$ and $\mathrm{Im}\,(p) = (2k+2)\pi$, for $k \in \mathbb{Z}$. Each J_k is mapped conformally to a copy of the λ-plane M_k. This map is not univalent though. However, it is univalent in the strip on J_0

$$\Gamma_p = \{p \in \mathbb{C} | 0 < \mathrm{Im}\,(p) \leqslant \pi\}.$$

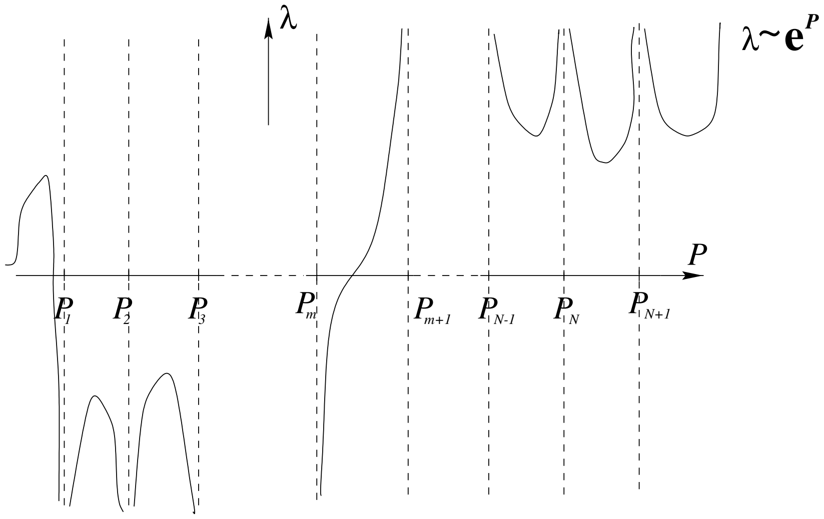

Figure 2. Graph of Re $(\lambda(p))$.

We know that $\lambda(p)$ has branch cuts along the segment of the real axis and that for $F_i \geqslant 0, i = 1, \ldots, N$ the graph of Re $(\lambda(p))$ is as in figure 2, while for $i = 1, \ldots, N$,

$$\mathrm{Im}\,(\lambda) = \begin{cases} -\pi F_i & p \in [p_i(x,t), p_{i+1}(x,t)] \\ 0 & \text{elsewhere.} \end{cases}$$

We conclude that the image of Γ_p is

$$\Gamma_\lambda = \left\{ \lambda \in \mathbb{C} | \mathrm{Im}\,(\lambda) \geqslant -\pi \sum_{k=1}^{N} F_k \right\} \subset M_0.$$

An example of this is illustrated in figure 3. The image of the strip Im $(p) \in [\pi, 2\pi)$ is the mirror image of this, by periodicity and the Schwarz reflection principle—the two image regions are joined along B′C′.

Lemma 4.1. *$\lambda(p)$, defined by (20), maps the strip Γ_p univalently and conformally to Γ_λ. Thus, for fixed x, an inverse function $p(\lambda)$ is well defined in the region $\Gamma_\lambda = \{\lambda \in \mathbb{C} | \mathrm{Im}\,(\lambda) \geqslant -\pi \sum_{k=1}^{N} F_k\}$. This inverse mapping is discontinuous along the union of horizontal slits $\Sigma = (\cup_{i=1}^{m}(-\infty, \hat{\lambda}_i]) \bigcup (\cup_{j=m+1}^{N+1}[\hat{\lambda}_j, +\infty))$, but analytic throughout domain $D = \Gamma_\lambda - \Sigma$.*

We seek a transformation from the canonical variables (x, p) to new canonical variables (η, λ), such that the new momentum λ is constant on the characteristics. A generating function S for such a transformation, defined by

$$S(x, \lambda) = \int_{x_0}^{x} p(x', \lambda, t)\, \mathrm{d}x' \qquad x_0 \in \mathbb{R} \tag{22}$$

was constructed [4]. We suppose that $p(x, \lambda, t)$ tends, as $x \to x_0$, to some x-independent asymptotic value $p_0(\lambda) = p(x_0, \lambda, t)$. The transformation is then defined implicitly by

$$p(x, \lambda) = \frac{\partial S}{\partial x} \tag{23}$$

$$\eta(x, \lambda) = \frac{\partial S}{\partial \lambda} \tag{24}$$

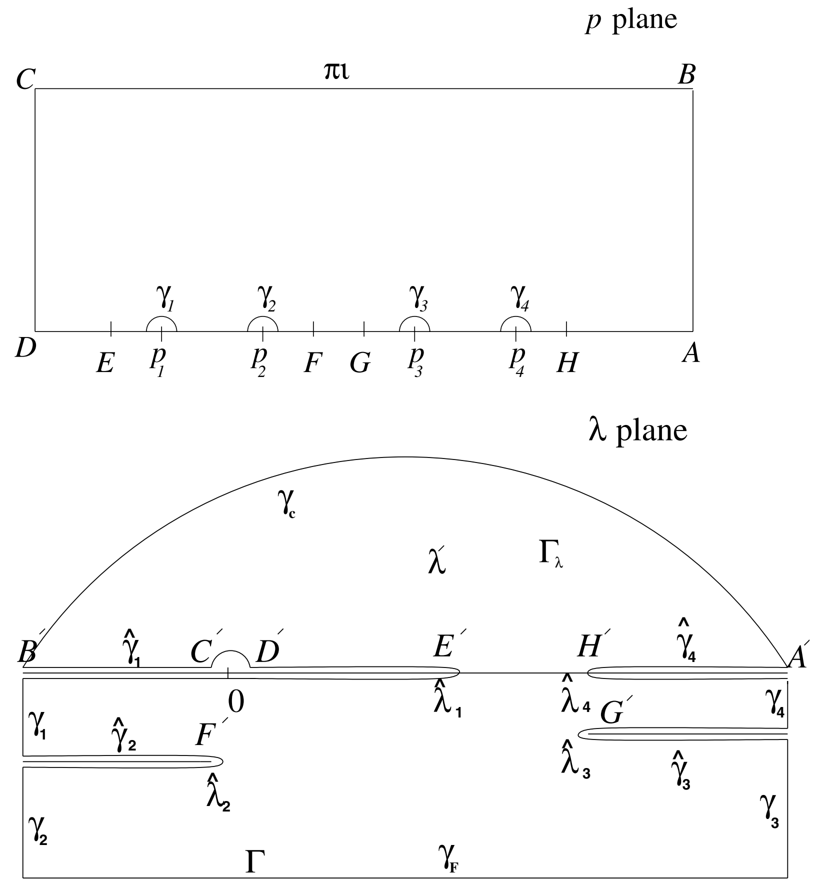

Figure 3. Example of p plane and contour Γ for the case $(2, 2)$.

while the new Hamiltonian is given by

$$K_n(\eta, \lambda) = H_n(x, p) + \frac{\partial S}{\partial t}. \tag{25}$$

The formula (22) is clearly just the x-integral of (23); further, since

$$\frac{\partial p}{\partial t_n} + \frac{\partial}{\partial x} H_n(p, x) = 0$$

we obtain $K_n(\eta, \lambda) = H_n(p_0(\lambda)) = H_n(p_0(\lambda(x, p)))$; the new coordinate η conjugate to λ is given by (24)

$$\eta = \frac{\partial S}{\partial \lambda} = \int_{x_0}^{x} \frac{\partial p}{\partial \lambda}\, \mathrm{d}x'.$$

With the choice of boundary conditions above, $K_n(\eta, \lambda)$ will be independent of η, that is, $K_n(\eta, \lambda) = K(\lambda)$. The canonical equations are generated by $K(\lambda)$ in the usual way†

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t_n} = -\frac{\partial K(\lambda)}{\partial \eta} = 0$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t_n} = \frac{\partial K(\lambda)}{\partial \lambda} = \text{constant}.$$

† For brevity, the time dependence of all functions will be omitted hereafter.

As a result of the construction, $p(x,\lambda)$ has $N+1$ branch points $\hat{\lambda}_k$, and the branch cuts are taken from $-\infty$ to $\hat{\lambda}_i$ and from $\hat{\lambda}_j$ to $+\infty$, where $i=1,\ldots,m$, $j=m+1,\ldots,N+1$. Since $p(x,\lambda)$ is a function of x, t_n as well as λ, the Riemann invariants are also functions of x and t_n. We now suppose that $\hat{\lambda}_i(x)$ are strictly monotonically increasing functions of x, while $\hat{\lambda}_j(x)$ are strictly monotonically decreasing functions of x, that is, we require

$$\frac{\partial\hat{\lambda}_i}{\partial x}>0 \qquad \text{and} \qquad \frac{\partial\hat{\lambda}_j}{\partial x}<0 \tag{26}$$

respectively. Now, $p(x,\lambda)$ is analytic everywhere inside the domain Γ_λ. On the boundary Γ, it is analytic on γ_c, and real on γ_i, $i=1,\ldots,N+1$ and γ_F. On the cuts $\hat{\gamma}_k$, it is either analytic for all x or there exists some unique value $x^*(\lambda)$, such that $\hat{\lambda}_k(x^*(\lambda))=\lambda$. Then for $x>x^*(\lambda)$, $p(x,\lambda)$ is real on $\hat{\gamma}_i$, $i\neq 1$ and on $\hat{\gamma}_1$ for $x>x^*(\lambda)$ with $\lambda>0$. At the branch points $\hat{\lambda}_k$, if $\frac{\partial^2\lambda}{\partial p^2}|_{p=\hat{p}_k}\neq 0$, $p(x,\lambda)$ has the behaviour

$$p=\hat{p}_k+\mathrm{O}((\lambda-\hat{\lambda}_k)^{\frac{1}{2}}) \tag{27}$$

since, at the turning points $\hat{p}_k$, Taylor's expansion gives

$$\lambda=\hat{\lambda}_k+\frac{1}{2}\frac{\partial^2\lambda}{\partial p^2}\bigg|_{p=\hat{p}_k}(p-\hat{p}_k)^2+\mathrm{O}((p-\hat{p}_k)^3).$$

Lemma 4.2. *If the Riemann invariants $\hat{\lambda}_i$ satisfy the monotonicity conditions above (equation (26)), then the region of analyticity of $p(x,\lambda)$, $D=D(x)$ as in lemma 4.1, satisfies $D(x_1)\subset D(x_2)$ for $x_1>x_2$.*

Corollary 4.1. *The region of analyticity of $S(x,\lambda)$ is*

$$\bigcap_{x_0\leqslant x'\leqslant x} D(x')=D(x).$$

Now on $\hat{\gamma}_i$, $i\neq 1$, for $x>x^*(\lambda)$, $p(x,\lambda)$ is real, and for $x>x^*(\lambda)$, with $\lambda>0$ on $\hat{\gamma}_1$, the imaginary part of $S(x,\lambda)$ is then given by

$$\mathrm{Im}\,(S)=\int_{x_0}^{x^*(\lambda)}\mathrm{Im}\,(p(x',\lambda))\,\mathrm{d}x'.$$

Integrating (27) with respect to x, at the branch points $\hat{\lambda}_k$, and using $\frac{\partial\hat{\lambda}_k}{\partial x}\neq 0$, an expansion of S has the form

$$S=S_k+\mathrm{O}((\lambda-\hat{\lambda}_k)^{\frac{3}{2}}). \tag{28}$$

Now as $|\lambda|\to\infty$, p has asymptotics $\ln\lambda+\mathrm{O}(\frac{1}{\lambda})$, and so $S\sim(x-x_0)\ln\lambda+\mathrm{O}(\frac{1}{\lambda})$ as $|\lambda|\to\infty$. Following [7], we define functions $\Omega(x,\lambda)$, $\Xi(x,\lambda)$ by

$$\Omega(x,\lambda)=S(x,\lambda)-K(\lambda)t_n \tag{29}$$

$$\Xi(x,\lambda)=\Omega(x,\lambda)+H_n(x,p)t_n-(x-x_0)p(x,\lambda). \tag{30}$$

Since as $|\lambda|\to\infty$, $\Omega\sim(x-x_0)\ln\lambda-K(\lambda)t_n+\mathrm{O}(\frac{1}{\lambda})$, it follows that $\Xi\sim\mathrm{O}(\frac{1}{\lambda})$ as $|\lambda|\to\infty$. Now $\mathrm{Im}\,(\Omega)$ is independent of time on the cuts, because

$$\begin{aligned}\frac{\partial\Omega}{\partial t_n}&=\frac{\partial S}{\partial t_n}-K(\lambda)\\&=-H_n(x,p).\end{aligned}$$

For $x>x^*(\lambda)$, $p(x,\lambda)$ is real on the cuts; $H_n(x,p)$ is real if p is, so we have $\frac{\partial\mathrm{Im}\,(\Omega(x,\lambda))}{\partial t_n}=0$. We also see that $\mathrm{Im}\,(\Xi(x,\lambda))=\mathrm{Im}\,(\Omega(x,\lambda))$ on the cut. Further, on the cuts $\hat{\gamma}_k$ for

$x > x^*(\lambda)$, $\Xi(x,\lambda)$ is real; we know that on $\hat{\gamma}_i$, $i \neq 1$, for $x > x^*(\lambda)$, $p(x,\lambda)$ is real and so is $\Xi(x,\lambda)$. Now consider the cut $(-\infty, \hat{\lambda}_1]$ for $\lambda < 0$ at $t_n = 0$, we see that $\mathrm{Im}\,(\Xi(x,\lambda))|_{\hat{\gamma}_1} = \mathrm{Im}\,(\Omega(x,\lambda)) - (x-x_0)\mathrm{Im}\,p(x,\lambda)$. This is zero since $\mathrm{Im}\,(\Omega(x,\lambda))|_{\hat{\gamma}_1} = \pi(x-x_0)$ and $\mathrm{Im}\,(p)|_{\hat{\gamma}_1} = \pi$ for $\lambda \in (-\infty, 0]$.

Theorem 4.1. *The solution of the reduced equations of motion*

$$\frac{\partial\lambda(\hat{p}_k)}{\partial t_n} + \mu_n(\hat{p}_k)\frac{\partial\lambda(\hat{p}_k)}{\partial x} = 0 \qquad k = 1,\ldots,N+1 \tag{31}$$

can be expressed in terms of the hodograph equations

$$x - x_0 - \mu_n(\hat{p}_k)t_n = -\frac{1}{\pi}\sum_{i=1}^{N+1} P\int_{\hat{\gamma}_i} \frac{\mathrm{e}^{\hat{p}_k}}{\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{\hat{p}_k}}\,\mathrm{d}\,(\mathrm{Im}\,(\Omega)) \tag{32}$$

with

$$\mathrm{Im}\,(\Omega)|_{t_n=0} = \int_{x_0}^{x^*(\lambda)} \mathrm{Im}\,(p(x',\lambda,0))\,\mathrm{d}x'.$$

Proof. Let $\partial D = \sum_{k=1}^{N+1}\hat{\gamma}_k + \sum_{i=1}^{N+1}\gamma_i + \gamma_F + \gamma_{\mathrm{c}}$ be our contour; by construction, Ξ is analytic inside Γ; we then consider the following integral:

$$\Xi(x,\lambda') = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma Q(\lambda,\lambda')\Xi(x,\lambda)\,\mathrm{d}\lambda$$

where

$$Q(\lambda,\lambda') = \frac{\mathrm{e}^{p(x,\lambda)}}{\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{p(x,\lambda')}}\frac{\partial p(x,\lambda)}{\partial\lambda}.$$

Letting λ' approach $\hat{\gamma}_k$, indenting the contour, and denoting $\sum_{i=1}^{N+1}\gamma_i + \gamma_F + \gamma_{\mathrm{c}}$ by Γ_γ, we obtain

$$\Xi(x,\lambda') = \frac{1}{\pi\mathrm{i}}\int_{\Gamma_\gamma} Q(\lambda,\lambda')\Xi(x,\lambda)\,\mathrm{d}\lambda + \frac{1}{\pi\mathrm{i}}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} Q(\lambda,\lambda')\Xi(x,\lambda)\,\mathrm{d}\lambda. \tag{33}$$

Collecting the real parts on both sides, equation (33) becomes

$$\begin{aligned}\mathrm{Re}\,(\Xi(x,\lambda')) &= \frac{1}{\pi}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} \mathrm{Re}\,(Q(\lambda,\lambda'))\mathrm{Im}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\int_{\Gamma_\gamma} \mathrm{Im}\,(Q(\lambda,\lambda'))\mathrm{Re}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda + \frac{1}{\pi}\int_{\Gamma_\gamma} \mathrm{Re}\,(Q(\lambda,\lambda'))\mathrm{Im}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} \mathrm{Im}\,(Q(\lambda,\lambda'))\mathrm{Re}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda.\end{aligned}$$

Since $p(x,\lambda)$ is real except on the curve γ_{c}, it follows that $Q(\lambda,\lambda')$ is real† on $\gamma_F + \sum_{i=1}^{N+1}\gamma_i + \sum_{k=1}^{N+1}\hat{\gamma}_k$. Now on γ_F, $p(x,\lambda)$ is always real for all x; $\Xi(x,\lambda)$ will also be real, so the integral vanishes here. On either γ_i or γ_j for $i = 1,\ldots,m$, $j = m+1,\ldots,N+1$, $\Xi(x,\lambda)$ is like

$$\Xi(x,\lambda)|_{\gamma_i} = \Omega(x,\lambda)|_{\gamma_i} - (x-x_0)p_i + H_n(x,p_i)t_n$$
$$\Xi(x,\lambda)|_{\gamma_j} = \Omega(x,\lambda)|_{\gamma_j} - (x-x_0)p_j + H_n(x,p_j)t_n$$

since, in the vicinity of p_i (p_j), p can be expressed as $p_i + \mathrm{O}(\exp\{+\lambda\})(p_j + \mathrm{O}(\exp\{-\lambda\}))$, so p behaves like $p_i(p_j)$ on $\gamma_i(\gamma_j)$. This implies that $\Xi(x,\lambda)$ is real on either γ_i or γ_j. On

† For $x > x^*$ on $\hat{\gamma}_k$, $k = 1,\ldots,N+1$.

the the arcs enclosing the cuts $\hat{\gamma}_k$, since $p(x,\lambda)$ is real for $x > x^*$, $Q(\lambda,\lambda')$ is also real, but $\Xi(x,\lambda)$ has a nonzero imaginary part in the intervals

$$[\hat{\lambda}_i(x_0), \hat{\lambda}_i(x^*)] \cup [\hat{\lambda}_j(x^*), \hat{\lambda}_j(x_0)].$$

Therefore, we obtain

$$\begin{aligned}\mathrm{Re}\,(\Xi(x,\lambda')) &= \frac{1}{\pi}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} Q(\lambda,\lambda')\mathrm{Im}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\int_{\gamma_c} \mathrm{Im}\,(Q(\lambda,\lambda'))\mathrm{Re}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda + \frac{1}{\pi}\int_{\gamma_c} \mathrm{Re}\,(Q(\lambda,\lambda'))\mathrm{Im}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda.\end{aligned}$$

Differentiating with respect to λ', we have

$$\begin{aligned}\frac{\partial \mathrm{Re}\,(\Xi(x,\lambda'))}{\partial\lambda'} &= \frac{1}{\pi}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} \frac{\partial Q(\lambda,\lambda')}{\partial\lambda'}\mathrm{Im}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\int_{\gamma_c} \frac{\partial \mathrm{Im}\,(Q(\lambda,\lambda'))}{\partial\lambda'}\mathrm{Re}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\int_{\gamma_c} \frac{\partial \mathrm{Re}\,(Q(\lambda,\lambda'))}{\partial\lambda'}\mathrm{Im}\,(\Xi(x,\lambda))\,\mathrm{d}\lambda.\end{aligned} \qquad (34)$$

Now since

$$Q(\lambda,\lambda')\,\mathrm{d}\lambda = \frac{\mathrm{e}^{p(x,\lambda)}}{\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{p(x,\lambda')}}\,\mathrm{d}p$$

and so

$$\begin{aligned}\frac{\partial Q(\lambda,\lambda')}{\partial\lambda'}\,\mathrm{d}\lambda &= \frac{\partial p(x,\lambda')}{\partial\lambda'}\frac{\mathrm{e}^{p(x,\lambda)+p(x,\lambda')}}{(\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{p(x,\lambda')})^2}\,\mathrm{d}p \\ &= \frac{\partial p(x,\lambda')}{\partial\lambda'}\frac{\partial}{\partial\lambda}\left(-\frac{\mathrm{e}^{p(x,\lambda')}}{\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{p(x,\lambda')}}\right)\mathrm{d}p\end{aligned}$$

equation (34) becomes

$$\begin{aligned}\frac{\partial \mathrm{Re}\,(\Xi(x,\lambda'))}{\partial\lambda'} &= \frac{1}{\pi}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} Q'(\lambda,\lambda')\frac{\partial \mathrm{Im}\,(\Xi(x,\lambda))}{\partial\lambda}\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\int_{\gamma_c} \mathrm{Im}\,(Q'(\lambda,\lambda'))\frac{\partial \mathrm{Re}\,(\Xi(x,\lambda))}{\partial\lambda}\,\mathrm{d}\lambda \\ &\quad + \frac{1}{\pi}\int_{\gamma_c} \mathrm{Re}\,(Q'(\lambda,\lambda'))\frac{\partial \mathrm{Im}\,(\Xi(x,\lambda))}{\partial\lambda}\,\mathrm{d}\lambda\end{aligned}$$

where

$$Q'(\lambda,\lambda') = \frac{\mathrm{e}^{p(x,\lambda')}}{\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{p(x,\lambda')}}\frac{\partial p(x,\lambda')}{\partial\lambda'}.$$

The integrals on γ_c must vanish as $Q'(\lambda,\lambda')$ and $\frac{\partial\Xi(x,\lambda)}{\partial\lambda}$ are both $\mathrm{O}(\frac{1}{\lambda})$ as $|\lambda| \to \infty$. Taking the fact that $\mathrm{Im}\,(\Xi) = \mathrm{Im}\,(\Omega)$, we obtain

$$\begin{aligned}&\frac{\partial\Omega(x,\lambda')}{\partial\lambda'} + \left\{\frac{\partial H_n(x,p(x,\lambda'))}{\partial p}t_n - (x - x_0)\right\}\frac{\partial p(x,\lambda')}{\partial\lambda'} \\ &\quad = \frac{1}{\pi}\sum_{k=1}^{N+1} P\int_{\hat{\gamma}_k} Q'(\lambda,\lambda')\,\mathrm{d}\,(\mathrm{Im}\,(\Omega)).\end{aligned} \qquad (35)$$

Write $\mu_n(p) = \frac{\partial H_n(x,p)}{\partial p}$, and let $\lambda' \to \hat{\lambda}_k$, $\frac{\partial \lambda'}{\partial p} \to 0$. Now $\frac{\partial \Omega}{\partial p}$ is bounded at $\hat{\lambda}_k$, since, by using equation (28), we see that Ω is like $\Omega_k + \mathrm{O}((\lambda - \hat{\lambda}_k)^{\frac{3}{2}})$ at the branch points. Moreover, $\frac{\partial \lambda}{\partial p} = \mathrm{O}((\lambda - \hat{\lambda}_k)^{\frac{1}{2}})$ near $\hat{\lambda}_k$. It follows that $\frac{\partial \Omega}{\partial \lambda'}$ is bounded as $\lambda \to \hat{\lambda}_k$, and hence $\frac{\partial \Omega}{\partial p'}$ must vanish at the branch points $\hat{\lambda}_k$. We finally obtain, on dividing (35) by $\frac{\partial p}{\partial \lambda}$ and evaluating at the branch points, the stated result

$$x - x_0 - \mu_n(\hat{p}_k)t_n = -\frac{1}{\pi}\sum_{i=1}^{N+1} P\int_{\hat{\gamma}_i} \frac{\mathrm{e}^{\hat{p}_k}}{\mathrm{e}^{p(x,\lambda)} - \mathrm{e}^{\hat{p}_k}}\,\mathrm{d}\,(\mathrm{Im}\,(\Omega)) \qquad i = 1, \ldots, N+1. \tag{36}$$

□

This is the solution of the initial value problem. The N-waterbag reduction of the dΔKP hierarchy is a system of hydrodynamic type with $N+1$ dependent variables. With our solution (36), we see that the left-hand side is in the form of Tsarev's hodograph solution (7), while the right-hand side contains an integral. The kernel $Q'(\lambda, \lambda')$ of this integral is the generating function for the characteristic speeds of commuting flows $\mu_m(\hat{p}_k)$, which solve an overdetermined system of partial differential equations equivalent to (8). This kernel is the same for any reduction of the hierarchy. We note here that the solution of this reduction is similar in structure to the solution of the dispersionless Toda equations given by Kodama [11], especially the kernel $Q'(\lambda, \lambda')$. However, the contours γ_i depend explicitly on the reduction; indeed, reductions can be parametrized by a choice of these contours. Finally, the measure d(Im (Ω)) depends explicitly on the initial data.

References

[1] Benney D J 1973 Some properties of long non-linear waves *Stud. Appl. Math.* **52** 45
[2] Flaschka H 1974 On Toda lattice II *Prog. Theor. Phys.* **51** 703
[3] Flaschka H 1974 The Toda lattice II. Existence of integrals *Phys. Rev.* B **9** 9
[4] Geogjaev V V 1985 Quasiclassical approximation in the inverse scattering problem. *Sov. Phys.–Dokl.* **30** 840
[5] Geogjaev V V 1994 The quasiclassical limit of the inverse scattering problem method *Singular Limits of Dispersive Waves (NATO ASI Series B 320)* ed N Ercolani *et al* (New York: Plenum) p 53
[6] Gibbons J 1981 Collisionless Boltzmann equations and integrable moment equations *Physica* D **3** 503
[7] Gibbons J and Kodama Y 1994 Solving dispersionless Lax equations *Singular Limits of Dispersive Waves (NATO ASI Series B 320)* ed N Ercolani *et al* (New York: Plenum) p 61
[8] Gibbons J and Kupershmidt B A 1991 Relativistic analogs of basic integrable systems *Integrable and Superintegrable System* ed B A Kupershmidt (Singapore: World Scientific)
[9] Gibbons J and Tsarev S P 1996 Reductions of Benney's equations *Phys. Lett.* A **211** 19
[10] Gibbons J and Tsarev S P 1999 Conformal maps and reductions of the Benney equations *Phys. Lett.* A **258** 263
[11] Kodama Y 1990 Solution of the dispersionless Toda equation *Phys. Lett.* A **147** 477
[12] Kupershmidt B A 1985 Discrete Lax equations and differential-difference calculus *Astérisque* **123** 212
[13] Toda M 1975 Studies of a nonlinear lattice *Phys. Rep.* **18c** 1
[14] Toda M 1989 *Nonlinear Waves and Solitons* (London: Kluwer)
[15] Toda M 1989 *Theory of Nonlinear Lattices* (Berlin: Springer)
[16] Tsarev S P 1985 On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type *Sov. Math.–Dokl.* **31** 488
[17] Whitham G B 1973 *Linear and Nonlinear Waves* (New York: Wiley)